Why Isn't There an Inverse Quotient Rule?

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$$\int \frac{xe^{2x}}{(1+2x)^2}\,dx$$

Outline

1. Motivating Example
2. Generalization of the Inverse Quotient Rule
3. Why Isn't This in Textbooks?
4. Conclusion

Source: Stewart, "Essential Calculus, Early Transcendentals, 2nd edition"

Section 6.1 (Integration by Parts) Exercise #15.

$$\int \frac{xe^{2x}}{(1+2x)^2}\,dx$$

This presentation is based on my paper of the same title, which will be published in the Pittsburgh Interdisciplinary Mathematics Review.

Integration by Parts

- This strategy comes from the Product Rule!

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Ex: $\displaystyle\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx\,.$

$$f = x \qquad g' = \cos(x)\,dx$$

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Section 6.1 (Integration by Parts) Exercise #15.

$$\int \frac{xe^{2x}}{(1+2x)^2}\,dx$$

LIATE

- What should we choose for f?

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- What should we choose for f?

Logarithms	$\ln(x)$
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Algebraic Functions*	x^2

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Logarithms	$\ln(x)$
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Algebraic Functions*	x^2
Trig	$\sin(x)$
Exponentials	2^x

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$\star \dfrac{x}{(1+2x)^2}$ is also an algebraic function!

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$$I = \int \frac{xe^{2x}}{(1+2x)^2}\,dx$$

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$$I = \int \frac{xe^{2x}}{(1+2x)^2}\,dx$$

$$f = \frac{x}{(1+2x)^2} \qquad g' = e^{2x}\,dx$$

$$f' = \frac{1-2x}{(1+2x)^3}\,dx \qquad g = \frac{1}{2}e^{2x}$$

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$$I = \frac{xe^{2x}}{2(1+2x)^2} - \int \frac{(1-2x)e^{2x}}{2(1+2x)^3}\,dx$$

Best Parts for Section 6.1, Exercise #15:

$$\int \frac{xe^{2x}}{(1+2x)^2}\,dx$$

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$$f = xe^{2x} \qquad g' = \frac{1}{(1+2x)^2}\, dx\,.$$

Questions:

Best Parts for Section 6.1, Exercise #15:

$$\int \frac{xe^{2x}}{(1+2x)^2}\, dx$$

$$f = xe^{2x} \qquad g' = \frac{1}{(1+2x)^2}\, dx\,.$$

Questions:

- Why?

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Questions:

- Why?
- How?

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Questions:

- Why?
- How?

Let's consider another angle...

The Inverse Quotient Rule

$$I = \int \frac{xe^{2x}}{(1+2x)^2}\,dx$$

- Let's look at the Quotient Rule:

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The Inverse Quotient Rule

$$I = \int \frac{xe^{2x}}{(1+2x)^2}\,dx$$

- Let's look at the Quotient Rule:

$$\left(\frac{p}{q}\right)' = \frac{qp' - pq'}{q^2}$$

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$$q = 1 + 2x$$

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$$I = \frac{-xe^{2x}}{2(1+2x)} + \int \frac{2xe^{2x} + e^{2x}}{2(1+2x)}\,dx$$

$$= \frac{-xe^{2x}}{2(1+2x)} + \int \frac{e^{2x}(2x+1)}{2(1+2x)}\,dx$$

The Inverse Quotient Rule

- Let's look at the Quotient Rule:

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$$I = \int \frac{xe^{2x}}{(1+2x)^2}\, dx$$

$$q = 1 + 2x \qquad q' = 2\, dx$$

$$p = \frac{1}{2}xe^{2x} \qquad p' = \frac{1}{2}(2xe^{2x} + e^{2x})\, dx$$

$$\begin{aligned} I &= \frac{-xe^{2x}}{2(1+2x)} + \int \frac{2xe^{2x} + e^{2x}}{2(1+2x)}\, dx \\ &= \frac{-xe^{2x}}{2(1+2x)} + \int \frac{e^{2x}(2x+1)}{2(1+2x)}\, dx \\ &= \frac{-xe^{2x}}{2(1+2x)} + \int \frac{1}{2}e^{2x}\, dx \end{aligned}$$

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The Parts are Clear!

Integration by Parts	Inverse Quotient Rule
$\displaystyle\int \frac{xe^{2x}}{(1+2x)^2}\,dx$	$\displaystyle\int \frac{xe^{2x}}{(1+2x)^2}\,dx$
$f = xe^{2x} \qquad g' = \dfrac{1}{(1+2x)^2}\,dx$ $f' = 2xe^{2x} + e^{2x}\,dx \qquad g = \dfrac{-1}{2(1+2x)}$	$q = 1+2x \qquad q' = 2\,dx$ $p = \dfrac{1}{2}xe^{2x} \qquad p' = \dfrac{1}{2}(2xe^{2x} + e^{2x})\,dx$
$\displaystyle = fg - \int gf'$ $\displaystyle = \frac{-xe^{2x}}{2(1+2x)} + \int \frac{2xe^{2x}+e^{2x}}{2(1+2x)}\,dx$	$\displaystyle = \frac{-p}{q} + \int \frac{p'}{q}$ $\displaystyle = \frac{-xe^{2x}}{2(1+2x)} + \int \frac{2xe^{2x}+e^{2x}}{2(1+2x)}\,dx$

What if the Denominator Isn't Squared?

$$\int \frac{\sin^4(x)}{\cos^3(x)}\,dx$$

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$$\int \frac{\cos(x)\sin^4(x)}{\cos^4(x)}\,dx$$

$$q = \cos^2(x)$$
$$q' = -\,2\cos(x)\sin(x)\,dx$$

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$$\int \frac{\sin^4(x)}{\cos^3(x)}\,dx$$

$$\int \frac{\cos(x)\sin^4(x)}{\cos^4(x)}\,dx$$

$$q = \cos^2(x)$$
$$q' = -2\cos(x)\sin(x)\,dx$$
$$p = \frac{-1}{2}\sin^3(x)$$

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$$\begin{aligned} q &= \cos^2(x) \\ q' &= -2\cos(x)\sin(x)\,dx \\ p &= \frac{-1}{2}\sin^3(x) \\ p' &= \frac{-3}{2}\sin^2(x)\cos(x)\,dx \end{aligned}$$

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$$\int \frac{\sin^4(x)}{\cos^3(x)}\,dx = -\frac{-\sin^3(x)}{2\cos^2(x)} + \int \frac{-3\sin^2(x)\cos(x)}{2\cos^2(x)}\,dx$$

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$$= \frac{\sin^3(x)}{2\cos^2(x)} - \frac{3}{2}\int \frac{\sin^2(x)}{\cos(x)}\,dx$$

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$$p' = \frac{-3}{2}\sin^2(x)\cos(x)\, dx$$

$$\begin{aligned}
\int \frac{\sin^4(x)}{\cos^3(x)}\, dx &= -\frac{-\sin^3(x)}{2\cos^2(x)} + \int \frac{-3\sin^2(x)\cos(x)}{2\cos^2(x)}\, dx \\
&= \frac{\sin^3(x)}{2\cos^2(x)} - \frac{3}{2}\int \frac{\sin^2(x)}{\cos(x)}\, dx \\
&= \frac{\sin^3(x)}{2\cos^2(x)} + \frac{3}{2}\int \frac{\cos^2(x) - 1}{\cos(x)}\, dx
\end{aligned}$$

What if the Denominator Isn't Squared?

$$\int \frac{\sin^4(x)}{\cos^3(x)}\,dx$$

$$\int \frac{\cos(x)\sin^4(x)}{\cos^4(x)}\,dx$$

$$q = \cos^2(x)$$
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&= \frac{\sin^3(x)}{2\cos^2(x)} + \frac{3}{2}\int \frac{\cos^2(x)-1}{\cos(x)}\,dx \\
&= \frac{\sin^3(x)}{2\cos^2(x)} + \frac{3}{2}\int \cos(x)\,dx - \frac{3}{2}\int \sec(x)\,dx
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&= \frac{\sin^3(x)}{2\cos^2(x)} + \frac{3}{2}\sin(x) - \frac{3}{2}\ln|\sec(x)+\tan(x)| + C.
\end{aligned}$$

A More General Formula!

$$\int \frac{PQ'}{Q^m}\,dx$$

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A More General Formula!

$$\int \frac{PQ'}{Q^m}\,dx$$
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$$= \boldsymbol{\frac{1}{m-1} \cdot \frac{-P}{Q^{m-1}} + \frac{1}{m-1}\int \frac{P'}{Q^{m-1}}\, dx\, .}$$

Why Isn't This in Textbooks?

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1. We can *always* use Integration by Parts...

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Theorem (Inverse Quotient Rule)

If we can write an integral in the form $\int \frac{pq'}{q^2}$ *for some differentiable functions* p *and* q*, then we may write* $\int \frac{pq'}{q^2} = \frac{-p}{q} + \int \frac{p'}{q}$.

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$$\begin{aligned} f &= p \qquad & g' &= \frac{q'}{q^2} \\ f' &= p' \qquad & g &= \frac{-1}{q}. \end{aligned}$$

$\square$

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$$\int \frac{pq'}{q^2} = p \cdot \frac{-1}{q} - \int p' \cdot \frac{-1}{q}$$
$$= \frac{-p}{q} + \int \frac{p'}{q}.$$

□

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Can We Actually Get Easier Integrals?

$$\int x^2 \cos(x)\, dx = x^2 \sin(x) - \int 2x \sin(x)\, dx$$

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- We jump from the harder integral to the easier one!

$$\int \frac{\sin(x)}{x^2}\, dx = \frac{-\sin(x)}{x} + {\color{red}\int \frac{\cos(x)}{x}\, dx}$$

- Uh-oh!

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Many Inconvenient Quotients

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Many simple-looking fractions do not have elementary anti-derivatives!

Textbook-Style Exercises

- Waiting for publication in a new undergraduate-run journal, *Pittsburgh Interdisciplinary Mathematics Review*.
- Includes 26 exercises for interested Integral Calculus students.
- Reach out if you'd like me to send the paper/exercise list! ERH129@pitt.edu

Thank You!

- James Stewart for providing the motivating integral,
- Steve Chow a.k.a blackpenredpen for his solution to the motivating integral,
- Lark Song and the *Pittsburgh Interdisciplinary Mathematics Review* team for motivating me to pursue this topic,
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