Tensor Products $\bigotimes$ 1-Norms $\bigotimes$ Asymptotics

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Outline

1. The Problem
   - Tensors
   - Combinatorics
   - Norms
   - Asymptotics
   - Quick Calculations
2. Solution
   - Polynomial Generating Functions
   - 1-Norm of the Coefficients
   - Asymptotics

The Problem

"Counting with Tensor Products"

Asked by squiggles and answered by Gjergji Zaimi on August 27, 2017:

Suppose we have vectors $\boldsymbol{v} = (1, -1)$ and $\boldsymbol{w} = (1, 1)$ and any $m \in \mathbb{N}$. Let $a_m = \boldsymbol{v} \bigotimes \boldsymbol{v} \bigotimes \boldsymbol{w}^{\bigotimes m}$, and let $\tilde{a}_m$ be the sum over all $\binom{m+2}{2}$ unique vectors obtained by permuting the tensor coordinates of a_m. What are the asymptotics of [the] function $f(m) = ||\tilde{a}_m||_1$, where $||.||_1$ is the usual 1-norm in terms of m?

Inner Product

Definition (Inner Product)

Let $\boldsymbol{a} = (a_1, \ldots, a_n)^\top$, $\boldsymbol{b} = (b_1, \ldots, b_n)^\top$ be column vectors in $\mathbb{R}^n$. We define the **inner product** (a.k.a. dot product) as $\boldsymbol{a} \cdot \boldsymbol{b} := \boldsymbol{a}^\top \boldsymbol{b} \in \mathbb{R}$.

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$\boldsymbol{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, and $\boldsymbol{a} \cdot \boldsymbol{b}$

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Recall: A row vector is a $1 \times n$ matrix, so we can multiply it by a $n \times 1$ column matrix to get a 1×1 matrix.

Cross Product

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Let $\boldsymbol{a} = (a_1, a_2, a_3), \boldsymbol{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$. We define the **cross product** as

$$\boldsymbol{a} \times \boldsymbol{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1) \in \mathbb{R}^3.$$

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$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

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Note: The vector $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$.

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Let $\boldsymbol{a} = (a_1, \ldots, a_n)^\top \in \mathbb{R}^n$, $\boldsymbol{b} = (b_1, \ldots, b_m)^\top \in \mathbb{R}^m$ be column vectors. We define the **outer product** (a.k.a. tensor product) as $\boldsymbol{a} \bigotimes \boldsymbol{b} := \boldsymbol{a}\boldsymbol{b}^\top \in \mathbb{R}^{n \times m}$.

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Recall: A column vector is a $n \times 1$ matrix, so we can multiply it by a $1 \times m$ row matrix to get a $n \times m$ matrix.

Question: What might it look like to compute $\boldsymbol{a}^{\bigotimes 3} = \boldsymbol{a} \bigotimes \boldsymbol{a} \bigotimes \boldsymbol{a}$?

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Idea: The tensor product $\boldsymbol{a} \bigotimes \boldsymbol{b}$ is every product of an element in $\boldsymbol{a}$ with an element in $\boldsymbol{b}$.

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$$= (\mathbf{-8, 12, 12, -18, 12, -18, -18, 27})^\top.$$

Recap: Tensor Product

- There are many ways to multiply vectors!
- Think of the **tensor product** $\boldsymbol{a} \bigotimes \boldsymbol{b}$ as every entry in $\boldsymbol{a}$ multiplied by every entry in $\boldsymbol{b}$.
- When tensoring many vectors together, it helps to write our result as one big vector (Kronecker product).

Permutations

Question: How many different orders can the vectors $\boldsymbol{a} = (-2, 3)^\top$, $\boldsymbol{b} = (4, 5)^\top$, $\boldsymbol{c} = (1, -1)^\top$ be tensored together?

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Answer: $m+2$ places to put one copy of $\boldsymbol{v}$, leaving $m+1$ places to put the second $\boldsymbol{v}$. The rest will be $\boldsymbol{w}$'s.

$$\text{Total: } \frac{(m+2)(m+1)}{2} = \frac{(m+2)!}{m!2!} = \binom{m+2}{2} \text{ options.}$$

The Norm is a Distance

Definition (Norm)

Let $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ be a vector. The **norm of $\boldsymbol{a}$**, written $||\boldsymbol{a}||$, is the distance from $\boldsymbol{a}$ to the origin.

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For $\boldsymbol{a} = (-2, 3)$, $||a|| = \sqrt{(-2-0)^2 + (3-0)^2} = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$.

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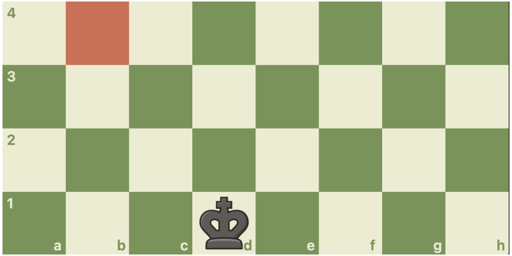

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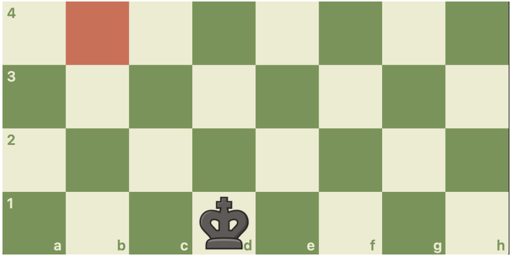
- How far away is the King from b4?
- If we're standing at the corner of 42nd St and 3rd Ave in a city, how far away are we from the corner of 40th St and 6th Ave?



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Remark: It can be shown for all $\boldsymbol{a} \in \mathbb{R}^n$ that

$$\lim_{p\to\infty} \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} = ||\boldsymbol{a}||_\infty.$$

Recap: Norms

- The norm of a vector is its distance from 0.
- Different notions of distance give rise to different norms.
- The 1-norm of $\boldsymbol{a} \in \mathbb{R}^n$ is $||\boldsymbol{a}||_1 = \sum_{i=1}^{n} |a_i|$.

Asymptotics

Definition (Asymptotic Equivalence)

We say that two functions $f(n)$, $g(n)$ are **asymptotically equivalent** and write $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.

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$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

Remark: $\sim$ is an equivalence relation.

“Counting with Tensor Products”

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Quick Calculations

When $m = 0$, we have

$$\tilde{a}_0 = \boldsymbol{v} \bigotimes \boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix},$$

so $||\tilde{a}_0||_1 = |1| + |-1| + |-1| + |1| = 4.$

Quick Calculations $(m = 1)$

$$||\tilde{a}_1||_1 = \left|\left|\boldsymbol{v} \bigotimes \boldsymbol{v} \bigotimes \boldsymbol{w} + \boldsymbol{v} \bigotimes \boldsymbol{w} \bigotimes \boldsymbol{v} + \boldsymbol{w} \bigotimes \boldsymbol{v} \bigotimes \boldsymbol{v}\right|\right|_1$$

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$$\begin{aligned} ||\tilde{a}_1||_1 &= \left|\left| \boldsymbol{v} \bigotimes \boldsymbol{v} \bigotimes \boldsymbol{w} + \boldsymbol{v} \bigotimes \boldsymbol{w} \bigotimes \boldsymbol{v} + \boldsymbol{w} \bigotimes \boldsymbol{v} \bigotimes \boldsymbol{v} \right|\right|_1 \\ &= \left|\left| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \bigotimes \boldsymbol{w} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \bigotimes \boldsymbol{v} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \bigotimes \boldsymbol{v} \right|\right|_1 \end{aligned}$$

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$$
\begin{aligned}
||\tilde{a}_1||_1 &= \left\| \boldsymbol{v} \bigotimes \boldsymbol{v} \bigotimes \boldsymbol{w} + \boldsymbol{v} \bigotimes \boldsymbol{w} \bigotimes \boldsymbol{v} + \boldsymbol{w} \bigotimes \boldsymbol{v} \bigotimes \boldsymbol{v} \right\|_1 \\
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&= \left\| \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\|
\end{aligned}
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$$
\begin{aligned}
||\tilde{a}_1||_1 &= \left\| \boldsymbol{v} \bigotimes \boldsymbol{v} \bigotimes \boldsymbol{w} + \boldsymbol{v} \bigotimes \boldsymbol{w} \bigotimes \boldsymbol{v} + \boldsymbol{w} \bigotimes \boldsymbol{v} \bigotimes \boldsymbol{v} \right\|_1 \\
&= \left\| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \bigotimes \boldsymbol{w} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \bigotimes \boldsymbol{v} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \bigotimes \boldsymbol{v} \right\|_1 \\
&= \left\| \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \right\|_1 \\
&= \left\| \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\|_1
\end{aligned}
$$

Quick Calculations $(m = 1)$

$$
\begin{aligned}
||\tilde{a}_1||_1 &= \left\| \boldsymbol{v} \bigotimes \boldsymbol{v} \bigotimes \boldsymbol{w} + \boldsymbol{v} \bigotimes \boldsymbol{w} \bigotimes \boldsymbol{v} + \boldsymbol{w} \bigotimes \boldsymbol{v} \bigotimes \boldsymbol{v} \right\|_1 \\
&= \left\| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \bigotimes \boldsymbol{w} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \bigotimes \boldsymbol{v} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \bigotimes \boldsymbol{v} \right\|_1 \\
&= \left\| \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \right\|_1 \\
&= \left\| \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\|_1 = \left\| \begin{pmatrix} 3 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \end{pmatrix} \right\|_1
\end{aligned}
$$

Quick Calculations $(m=1)$

$$\|\tilde{a}_1\|_1 = \left\| \boldsymbol{v} \bigotimes \boldsymbol{v} \bigotimes \boldsymbol{w} + \boldsymbol{v} \bigotimes \boldsymbol{w} \bigotimes \boldsymbol{v} + \boldsymbol{w} \bigotimes \boldsymbol{v} \bigotimes \boldsymbol{v} \right\|_1$$

$$= \left\| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \bigotimes \boldsymbol{w} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \bigotimes \boldsymbol{v} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \bigotimes \boldsymbol{v} \right\|_1$$

$$= \left\| \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \right\|_1$$

$$= \left\| \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\|_1 = \left\| \begin{pmatrix} 3 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \end{pmatrix} \right\|_1 = 12.$$

Quick Calculations ($m = 2$)

$$||\tilde{a}_2||_1 = \left\| \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\|_1$$

Quick Calculations $(m = 2)$

$$||\tilde{a}_2||_1 = \left\| \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\|_1 = \left\| \begin{pmatrix} 6 \\ 0 \\ 0 \\ -2 \\ 0 \\ -2 \\ -2 \\ 0 \\ 0 \\ -2 \\ -2 \\ 0 \\ -2 \\ 0 \\ 0 \\ 6 \end{pmatrix} \right\|_1$$

Quick Calculations ($m = 2$)

$$||\tilde{a}_2||_1 = \left\| \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\|_1 = \left\| \begin{pmatrix} 6 \\ 0 \\ 0 \\ -2 \\ 0 \\ -2 \\ -2 \\ 0 \\ 0 \\ -2 \\ -2 \\ 0 \\ -2 \\ 0 \\ 0 \\ 6 \end{pmatrix} \right\|_1$$

Quick Calculations ($m = 2$)

$$||\tilde{a}_2||_1 = \left\| \begin{pmatrix} 1\\1\\1\\1\\-1\\-1\\-1\\-1\\-1\\-1\\-1\\-1\\1\\1\\1\\1 \end{pmatrix} + \begin{pmatrix} 1\\1\\-1\\-1\\1\\1\\-1\\-1\\-1\\-1\\1\\1\\-1\\-1\\1\\1 \end{pmatrix} + \begin{pmatrix} 1\\-1\\1\\-1\\1\\-1\\1\\-1\\-1\\1\\-1\\1\\-1\\1\\-1\\1 \end{pmatrix} + \begin{pmatrix} 1\\1\\-1\\-1\\-1\\-1\\1\\1\\1\\1\\-1\\-1\\-1\\-1\\1\\1 \end{pmatrix} + \begin{pmatrix} 1\\-1\\1\\-1\\-1\\1\\-1\\1\\1\\-1\\1\\-1\\-1\\1\\-1\\1 \end{pmatrix} + \begin{pmatrix} 1\\-1\\-1\\1\\1\\-1\\-1\\1\\1\\-1\\-1\\1\\1\\-1\\-1\\1 \end{pmatrix} \right\|_1 = \left\| \begin{pmatrix} 6\\0\\0\\-2\\0\\-2\\-2\\0\\0\\-2\\-2\\0\\-2\\0\\0\\6 \end{pmatrix} \right\|_1 = 24.$$

Solution

Tensors vs. Polynomials

Compare these two operations:

Tensor Product:

$$(a, b) \bigotimes (c, d)$$
$$= \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix}$$
$$= \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

Tensors vs. Polynomials

Compare these two operations:

Tensor Product:

$$(a, b) \bigotimes (c, d)$$

$$= \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix}$$

$$= \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

Polynomial Product:

$$(a + b) \cdot (c + d)$$

$$= ac + ad + bc + bd$$

Tensors vs. Polynomials

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$$(a, b) \bigotimes (c, d)$$
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Polynomial Product:

$$(a + b) \cdot (c + d)$$
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This comparison extends to multiplication of several terms!

Tensors vs. Polynomials

If we don't combine terms on the right-hand side, this analogy will still hold!

Tensors vs. Polynomials

If we don't combine terms on the right-hand side, this analogy will still hold!

Tensor Product:

$$(1, x_1) \bigotimes (1, x_2)$$

$$= \begin{pmatrix} 1 \\ x_1 \end{pmatrix} \begin{pmatrix} 1 & x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ x_2 \\ x_1 \\ x_1 x_2 \end{pmatrix}$$

Polynomial Product:

$$(1 + x_1) \cdot (1 + x_2)$$

$$= 1 + x_2 + x_1 + x_1 x_2$$

Tensors vs. Polynomials

If we don't combine terms on the right-hand side, this analogy will still hold!

Tensor Product:

$$
\begin{aligned}
&(1, x_1) \bigotimes (1, x_2) \\
&= \begin{pmatrix} 1 \\ x_1 \end{pmatrix} \begin{pmatrix} 1 & x_2 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ x_2 \\ x_1 \\ x_1 x_2 \end{pmatrix}
\end{aligned}
$$

Polynomial Product:

$$
\begin{aligned}
&(1 + x_1) \cdot (1 + x_2) \\
&= 1 + x_2 + x_1 + x_1 x_2
\end{aligned}
$$

We can read the vector coordinates in each tensor product as the coefficients of a polynomial!

Polynomial Generating Functions

For each m, the entries of $\tilde{a}_m$ are the coefficients of

$$\sum_{\varepsilon} \prod_{i=1}^{m+2} (1 + \varepsilon_i x_i),$$

where two ε_i's are -1 and the rest are 1, for every possible ordering of the ε_i's.

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Example $(m = 1)$

$$\sum_{\varepsilon} \prod_{i=1}^{3} (1 + \varepsilon_i x_i)$$

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where two ε_i's are -1 and the rest are 1, for every possible ordering of the ε_i's.

Example $(m = 1)$

$$\sum_{\varepsilon} \prod_{i=1}^{3} (1 + \varepsilon_i x_i)$$
$$=(1 - x_1)(1 - x_2)(1 + x_3) + (1 - x_1)(1 + x_2)(1 - x_3) + (1 + x_1)(1 - x_2)(1 - x_3)$$

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where two ε_i's are -1 and the rest are 1, for every possible ordering of the ε_i's.

Example $(m = 1)$

$$\begin{aligned}
&\sum_{\varepsilon} \prod_{i=1}^{3} (1 + \varepsilon_i x_i) \\
=&(1 - x_1)(1 - x_2)(1 + x_3) + (1 - x_1)(1 + x_2)(1 - x_3) + (1 + x_1)(1 - x_2)(1 - x_3) \\
=&\mathbf{3} - \mathbf{1}x_1 - \mathbf{1}x_2 - \mathbf{1}x_3 - \mathbf{1}x_1x_2 - \mathbf{1}x_1x_3 - \mathbf{1}x_2x_3 + \mathbf{3}x_1x_2x_3.
\end{aligned}$$

Polynomial Generating Functions

Example ($m = 1$)

$$\sum_{\varepsilon} \prod_{i=1}^{3} (1 + \varepsilon_i x_i)$$
$$= (1 - x_1)(1 - x_2)(1 + x_3) + (1 - x_1)(1 + x_2)(1 - x_3) + (1 + x_1)(1 - x_2)(1 - x_3)$$
$$= \mathbf{3} - \mathbf{1}x_1 - \mathbf{1}x_2 - \mathbf{1}x_3 - \mathbf{1}x_1x_2 - \mathbf{1}x_1x_3 - \mathbf{1}x_2x_3 + \mathbf{3}x_1x_2x_3.$$

Note that we have a symmetric polynomial. Since we want the 1-norm (absolute value sum) of the vector of these coefficients, it will not matter if we add them all together!

Polynomial Generating Functions

Example $(m = 1)$

$$
\begin{aligned}
&\sum_{\varepsilon} \prod_{i=1}^{3} (1 + \varepsilon_i t) \\
=&(1-t)(1-t)(1+t) + (1-t)(1+t)(1-t) + (1+t)(1-t)(1-t) \\
=&\mathbf{3} - \mathbf{1}t - \mathbf{1}t - \mathbf{1}t - \mathbf{1}t^2 - \mathbf{1}t^2 - \mathbf{1}t^2 + \mathbf{3}t^3.
\end{aligned}
$$

Polynomial Generating Functions

Example $(m = 1)$

$$
\begin{aligned}
&\sum_{\varepsilon} \prod_{i=1}^{3} (1 + \varepsilon_i t) \\
=&(1-t)(1-t)(1+t) + (1-t)(1+t)(1-t) + (1+t)(1-t)(1-t) \\
=&\binom{3}{2}(1-t)^2(1+t).
\end{aligned}
$$

Polynomial Generating Functions

Example $(m = 1)$

$$
\begin{aligned}
&\sum_{\varepsilon} \prod_{i=1}^{3} (1 + \varepsilon_i t) \\
=&(1-t)(1-t)(1+t) + (1-t)(1+t)(1-t) + (1+t)(1-t)(1-t) \\
=&\binom{3}{2}(1-t)^2(1+t).
\end{aligned}
$$

The general formula is

$$
\binom{m+2}{2}(1-t)^2(1+t)^m.
$$

Recap: Polynomial Generating Functions

- The entries of $\tilde{a}_m$ are the coefficients of $\displaystyle\sum_{\varepsilon}\prod_{i=1}^{m+2}(1+\varepsilon_i x_i)$.
  - Two ε_i's are -1 and the rest are 1, and we're adding all $\binom{m+2}{2}$ ways of assigning the ε_i's.
- Since we want to compute $||\tilde{a}_m||_1$, we want the sum of the absolute values of all these coefficients.
- The resulting polynomial is always symmetric, so adding up the absolute values of the coefficients of $\displaystyle\sum_{\varepsilon}\prod_{i=1}^{m+2}(1+\varepsilon_i t) = \binom{m+2}{2}(1-t)^2(1+t)^m$ gives the same answer.
- So, if we can find a formula for each coefficient in this polynomial (and determine whether it's positive or negative), we can come up with a formula for $||\tilde{a}_m||_1$.

Add up the Coefficients

Our polynomial is $\binom{m+2}{2}(1-t)^2(1+t)^m$.

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Our polynomial is $\binom{m+2}{2}(1-t)^2(1+t)^m$. For each k, the coefficient of t^k is $\binom{m+2}{2}\left[\binom{m}{k} - 2\binom{m}{k-1} + \binom{m}{k-2}\right]$.

Add up the Coefficients

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$$\binom{m+2}{2}\begin{bmatrix} \binom{m}{0} & -2\binom{m}{-1} & +\binom{m}{-2} \\ +\binom{m}{1} & -2\binom{m}{0} & +\binom{m}{-1} \\ +\binom{m}{2} & -2\binom{m}{1} & +\binom{m}{0} \\ +\dots & -\dots & +\dots \\ +\binom{m}{m} & -2\binom{m}{m-1} & +\binom{m}{m-2} \\ +\binom{m}{m+1} & -2\binom{m}{m} & +\binom{m}{m-1} \\ +\binom{m}{m+2} & -2\binom{m}{m+1} & +\binom{m}{m} \end{bmatrix}$$

Add up the Coefficients

Our polynomial is $\binom{m+2}{2}(1-t)^2(1+t)^m$. For each k, the coefficient of t^k is $\binom{m+2}{2}\left[\binom{m}{k} - 2\binom{m}{k-1} + \binom{m}{k-2}\right]$. Let's add all the coefficients:

$$\binom{m+2}{2}\begin{bmatrix} \binom{m}{0} & -2\binom{m}{-1} & +\binom{m}{-2} \\ +\binom{m}{1} & -2\binom{m}{0} & +\binom{m}{-1} \\ +\binom{m}{2} & -2\binom{m}{1} & +\binom{m}{0} \\ +\dots & -\dots & +\dots \\ +\binom{m}{m} & -2\binom{m}{m-1} & +\binom{m}{m-2} \\ +\binom{m}{m+1} & -2\binom{m}{m} & +\binom{m}{m-1} \\ +\binom{m}{m+2} & -2\binom{m}{m+1} & +\binom{m}{m} \end{bmatrix}$$

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Our polynomial is $\binom{m+2}{2}(1-t)^2(1+t)^m$. For each k, the coefficient of t^k is $\binom{m+2}{2}\left[\binom{m}{k} - 2\binom{m}{k-1} + \binom{m}{k-2}\right]$. Let's add all the coefficients:

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Our polynomial is $\binom{m+2}{2}(1-t)^2(1+t)^m$. For each k, the coefficient of t^k is $\binom{m+2}{2}\left[\binom{m}{k} - 2\binom{m}{k-1} + \binom{m}{k-2}\right]$. Let's add all the coefficients:

$$\binom{m+2}{2}\begin{bmatrix} \binom{m}{0} & -2\binom{m}{-1} & +\binom{m}{-2} \\ +\binom{m}{1} & -2\binom{m}{0} & +\binom{m}{-1} \\ +\binom{m}{2} & -2\binom{m}{1} & +\binom{m}{0} \\ +\dots & -\dots & +\dots \\ +\binom{m}{m} & -2\binom{m}{m-1} & +\binom{m}{m-2} \\ +\binom{m}{m+1} & -2\binom{m}{m} & +\binom{m}{m-1} \\ +\binom{m}{m+2} & -2\binom{m}{m+1} & +\binom{m}{m} \end{bmatrix} = 0.$$

So, add up all the positive (or negative) coefficients, then double the result!

Positive vs. Negative Coefficients

For each k, the coefficient of t^k becomes

$$\binom{m+2}{2}\left[\binom{m}{k} - 2\binom{m}{k-1} + \binom{m}{k-2}\right]$$

Positive vs. Negative Coefficients

For each k, the coefficient of t^k becomes

$$\binom{m+2}{2}\left[\binom{m}{k}-2\binom{m}{k-1}+\binom{m}{k-2}\right]$$

$$=\frac{1}{2}\binom{m+2}{k}\left[(m+1-k)(m+2-k)-2(m+2-k)k+k(k-1)\right]$$

Positive vs. Negative Coefficients

For each k, the coefficient of t^k becomes

$$\begin{aligned}
&\boldsymbol{\binom{m+2}{2}\left[\binom{m}{k} - 2\binom{m}{k-1} + \binom{m}{k-2}\right]}\\
=&\frac{1}{2}\binom{m+2}{k}\left[(m+1-k)(m+2-k) - 2(m+2-k)k + k(k-1)\right]\\
=&\frac{1}{2}\binom{m+2}{k}\left[(m-2k+2)^2 - 2 - m\right],
\end{aligned}$$

Positive vs. Negative Coefficients

For each k, the coefficient of t^k becomes

$$\boldsymbol{\binom{m+2}{2}\left[\binom{m}{k}-2\binom{m}{k-1}+\binom{m}{k-2}\right]}$$

$$=\frac{1}{2}\binom{m+2}{k}\left[(m+1-k)(m+2-k)-2(m+2-k)k+k(k-1)\right]$$

$$=\frac{1}{2}\binom{m+2}{k}\left[(m-2k+2)^2-2-m\right],$$

which is non-positive (negative or 0) if and only if

$$(m-2k+2)^2-2-m\leq 0$$

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$$=\frac{1}{2}\binom{m+2}{k}\left[(m+1-k)(m+2-k)-2(m+2-k)k+k(k-1)\right]$$

$$=\frac{1}{2}\binom{m+2}{k}\left[(m-2k+2)^2-2-m\right],$$

which is non-positive (negative or 0) if and only if

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$$|m-2k+2|\leq\sqrt{m+2}$$

Positive vs. Negative Coefficients

For each k, the coefficient of t^k becomes

$$\boldsymbol{\binom{m+2}{2}\left[\binom{m}{k}-2\binom{m}{k-1}+\binom{m}{k-2}\right]}$$
$$=\frac{1}{2}\binom{m+2}{k}\left[(m+1-k)(m+2-k)-2(m+2-k)k+k(k-1)\right]$$
$$=\frac{1}{2}\binom{m+2}{k}\left[(m-2k+2)^2-2-m\right],$$

which is non-positive (negative or 0) if and only if

$$(m-2k+2)^2-2-m\leq 0$$
$$|m-2k+2|\leq\sqrt{m+2}$$
$$\boldsymbol{\frac{m+2-\sqrt{m+2}}{2}\leq k\leq\frac{m+2+\sqrt{m+2}}{2}}.$$

Positive vs. Negative Coefficients

For each k, the coefficient of t^k becomes

$$\boldsymbol{\binom{m+2}{2}\left[\binom{m}{k} - 2\binom{m}{k-1} + \binom{m}{k-2}\right]}$$

$$= \frac{1}{2}\binom{m+2}{k}\left[(m+1-k)(m+2-k) - 2(m+2-k)k + k(k-1)\right]$$

$$= \frac{1}{2}\binom{m+2}{k}\left[(m-2k+2)^2 - 2 - m\right],$$

which is non-positive (negative or 0) if and only if

$$(m-2k+2)^2 - 2 - m \leq 0$$

$$|m-2k+2| \leq \sqrt{m+2}$$

$$\boldsymbol{\left\lceil \frac{m+2-\sqrt{m+2}}{2} \right\rceil \leq k \leq \left\lfloor \frac{m+2+\sqrt{m+2}}{2} \right\rfloor}.$$

Gather Negative Coefficients

$$f(m) = -\mathbf{2}\binom{m+2}{2} \sum_{\left\lceil \frac{m+2-\sqrt{m+2}}{2} \right\rceil \le k \le \left\lfloor \frac{m+2+\sqrt{m+2}}{2} \right\rfloor} \left[\binom{m}{k} - 2\binom{m}{k-1} + \binom{m}{k-2}\right]$$

Gather Negative Coefficients

$$f(m) = -\mathbf{2}\binom{m+2}{2} \sum_{\left\lceil \frac{m+2-\sqrt{m+2}}{2} \right\rceil \leq k \leq \left\lfloor \frac{m+2+\sqrt{m+2}}{2} \right\rfloor} \left[\binom{m}{k} - 2\binom{m}{k-1} + \binom{m}{k-2}\right]$$

Example $(m = 6)$

We get $\lceil 4-\sqrt{2} \rceil \leq k \leq \lfloor 4+\sqrt{2} \rfloor$, so the formula is

Gather Negative Coefficients

$$f(m) = -2\binom{m+2}{2} \sum_{\left\lceil \frac{m+2-\sqrt{m+2}}{2} \right\rceil \le k \le \left\lfloor \frac{m+2+\sqrt{m+2}}{2} \right\rfloor} \left[\binom{m}{k} - 2\binom{m}{k-1} + \binom{m}{k-2} \right]$$

Example ($m = 6$)

We get $\lceil 4-\sqrt{2} \rceil \le k \le \lfloor 4+\sqrt{2} \rfloor$, so the formula is

$$-2\binom{8}{2}\begin{pmatrix} \binom{6}{3} & -2\binom{6}{2} & +\binom{6}{1} \\ +\binom{6}{4} & -2\binom{6}{3} & +\binom{6}{2} \\ +\binom{6}{5} & -2\binom{6}{4} & +\binom{6}{3} \end{pmatrix}$$

Gather Negative Coefficients

$$f(m) = \color{orange}{-2}\binom{m+2}{2} \sum_{\left\lceil \frac{m+2-\sqrt{m+2}}{2} \right\rceil \le k \le \left\lfloor \frac{m+2+\sqrt{m+2}}{2} \right\rfloor} \left[\binom{m}{k} - 2\binom{m}{k-1} + \binom{m}{k-2} \right]$$

Example $(m = 6)$

We get $\lceil 4 - \sqrt{2} \rceil \le k \le \lfloor 4 + \sqrt{2} \rfloor$, so the formula is

$$\color{orange}{-2}\binom{8}{2} \begin{pmatrix} \color{blue}{\binom{6}{3}} & -2\binom{6}{2} & +\binom{6}{1} \\ +\binom{6}{4} & \color{blue}{-2\binom{6}{3}} & +\binom{6}{2} \\ +\binom{6}{5} & -2\binom{6}{4} & +\color{blue}{\binom{6}{3}} \end{pmatrix}$$

Gather Negative Coefficients

$$f(m) = -2\binom{m+2}{2} \sum_{\left\lceil \frac{m+2-\sqrt{m+2}}{2} \right\rceil \le k \le \left\lfloor \frac{m+2+\sqrt{m+2}}{2} \right\rfloor} \left[\binom{m}{k} - 2\binom{m}{k-1} + \binom{m}{k-2} \right]$$

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Final Formula

For generic m, we obtain

$$f(m) = -2\binom{m+2}{2} \sum_{\left\lceil \frac{m+2-\sqrt{m+2}}{2} \right\rceil \le k \le \left\lfloor \frac{m+2+\sqrt{m+2}}{2} \right\rfloor} \left[\binom{m}{k} - 2\binom{m}{k-1} + \binom{m}{k-2} \right]$$

Final Formula

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$$f(m) = -2\binom{m+2}{2} \sum_{\left\lceil \frac{m+2-\sqrt{m+2}}{2} \right\rceil \le k \le \left\lfloor \frac{m+2+\sqrt{m+2}}{2} \right\rfloor} \left[\binom{m}{k} - 2\binom{m}{k-1} + \binom{m}{k-2} \right]$$

$$= \boldsymbol{4\binom{m+2}{2} \left[\binom{m}{\left\lceil \frac{m+2-\sqrt{m+2}}{2} - 1 \right\rceil} - \binom{m}{\left\lceil \frac{m+2-\sqrt{m+2}}{2} - 2 \right\rceil} \right]}.$$

Asymptotics (Recall: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$)

Let $\# = \left\lceil \frac{m+2-\sqrt{m+2}}{2} - 1 \right\rceil$. Then $f(m) = 4\binom{m+2}{2}\left[\binom{m}{\#} - \binom{m}{\#-1}\right]$.

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$$f(m) = \frac{4(m+2)!}{m!2!}\left[\frac{m!}{(m-\#)!\#!} - \frac{m!}{(m-\#+1)!(\#-1)!}\right]$$

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$$
\begin{aligned}
f(m) =& \frac{4(m+2)!}{m!2!}\left[\frac{m!}{(m-\#)!\#!} - \frac{m!}{(m-\#+1)!(\#-1)!}\right] \\
=& \frac{2(m+2)!}{\#!(m-\#+1)!}\left[m-\#+1-\#\right]
\end{aligned}
$$

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Let $\# = \left\lceil \frac{m+2-\sqrt{m+2}}{2} - 1 \right\rceil$. Then $f(m) = 4\binom{m+2}{2}\left[\binom{m}{\#} - \binom{m}{\#-1}\right]$.

$$\begin{aligned}
f(m) &= \frac{4(m+2)!}{m!2!}\left[\frac{m!}{(m-\#)!\#!} - \frac{m!}{(m-\#+1)!(\#-1)!}\right] \\
&= \frac{2(m+2)!}{\#!(m-\#+1)!}\left[m-\#+1-\#\right] \\
&\sim \frac{2\sqrt{2\pi(m+2)}\left(\frac{m+2}{e}\right)^{m+2}}{\left[\sqrt{2\pi\#}\left(\frac{\#}{e}\right)^{\#}\right]\left[\sqrt{2\pi(m-\#+1)}\left(\frac{m-\#+1}{e}\right)^{m-\#+1}\right]}\left[m-2\#+1\right]
\end{aligned}$$

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=& 2\sqrt{\frac{(m+2)}{2\pi\#(m-\#+1)}}
\end{aligned}$$

Asymptotics (Recall: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$)

Let $\# = \left\lceil \frac{m+2-\sqrt{m+2}}{2} - 1 \right\rceil$. Then $f(m) = 4\binom{m+2}{2}\left[\binom{m}{\#} - \binom{m}{\#-1}\right]$.

$$\begin{aligned}
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\sim& \frac{2\sqrt{2\pi(m+2)}\left(\frac{m+2}{e}\right)^{m+2}}{\left[\sqrt{2\pi\#}\left(\frac{\#}{e}\right)^{\#}\right]\left[\sqrt{2\pi(m-\#+1)}\left(\frac{m-\#+1}{e}\right)^{m-\#+1}\right]}\left[m-2\#+1\right] \\
=& 2\sqrt{\frac{(m+2)}{2\pi\#(m-\#+1)}} \cdot \frac{(m+2)^{m+2}}{\#^{\#}(m-\#+1)^{m-\#+1}}
\end{aligned}$$

Asymptotics (Recall: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$)

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$$
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=& \frac{2(m+2)!}{\#!(m-\#+1)!}\left[m-\#+1-\#\right] \\
\sim& \frac{2\sqrt{2\pi(m+2)}\left(\frac{m+2}{e}\right)^{m+2}}{\left[\sqrt{2\pi\#}\left(\frac{\#}{e}\right)^{\#}\right]\left[\sqrt{2\pi(m-\#+1)}\left(\frac{m-\#+1}{e}\right)^{m-\#+1}\right]}\left[m-2\#+1\right] \\
=& 2\sqrt{\frac{(m+2)}{2\pi\#(m-\#+1)}} \cdot \frac{(m+2)^{m+2}}{\#^{\#}(m-\#+1)^{m-\#+1}} \cdot \frac{1}{e}
\end{aligned}
$$

Asymptotics (Recall: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$)

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$$
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&= 2\sqrt{\frac{(m+2)}{2\pi\#(m-\#+1)}} \cdot \frac{(m+2)^{m+2}}{\#^{\#}(m-\#+1)^{m-\#+1}} \cdot \frac{1}{e} \cdot \left[m-2\#+1\right]
\end{aligned}
$$

More Asymptotics (Recall: $\# = \left\lceil \frac{m+2-\sqrt{m+2}}{2} - 1 \right\rceil$)

Note: $\# \sim \frac{m}{2}$, $m - \# \sim \frac{m}{2}$, and $m - 2\# \sim \sqrt{m+2}$.

$$f(m) = 2\sqrt{\frac{(m+2)}{2\pi\#(m-\#+1)}} \cdot \frac{(m+2)^{m+2}}{\#^{\#}(m-\#+1)^{m-\#+1}} \cdot \frac{1}{e} \cdot [m - 2\# + 1]$$

More Asymptotics (Recall: $\# = \left\lceil \frac{m+2-\sqrt{m+2}}{2} - 1 \right\rceil$)

Note: $\# \sim \frac{m}{2}$, $m - \# \sim \frac{m}{2}$, and $m - 2\# \sim \sqrt{m+2}$.

$$\begin{aligned} f(m) =& 2\sqrt{\frac{(m+2)}{2\pi\#(m-\#+1)}} \cdot \frac{(m+2)^{m+2}}{\#^{\#}(m-\#+1)^{m-\#+1}} \cdot \frac{1}{e} \cdot [m - 2\# + 1] \\ \sim& 2\sqrt{\frac{(m+2)}{2\pi\frac{m}{2}\cdot\frac{m}{2}}} \cdot \frac{(m+2)^{m+2}}{\left(\frac{m}{2}\right)^{\frac{m}{2}}\left(\frac{m}{2}\right)^{\frac{m}{2}}} \cdot \frac{1}{e} \cdot \sqrt{m+2} \end{aligned}$$

More Asymptotics (Recall: $\# = \left\lceil \frac{m+2-\sqrt{m+2}}{2} - 1 \right\rceil$)

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\sim& 2\sqrt{\frac{(m+2)}{2\pi\frac{m}{2}\cdot\frac{m}{2}}} \cdot \frac{(m+2)^{m+2}}{\left(\frac{m}{2}\right)^{\frac{m}{2}}\left(\frac{m}{2}\right)^{\frac{m}{2}}} \cdot \frac{1}{e} \cdot \sqrt{m+2} \\
=& \frac{2}{e}\sqrt{\frac{1}{\pi\frac{m^2}{2}}} \cdot \frac{(m+2)^{m+3}}{\left(\frac{m}{2}\right)^{m}}
\end{aligned}
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Note: $\# \sim \frac{m}{2}$, $m - \# \sim \frac{m}{2}$, and $m - 2\# \sim \sqrt{m+2}$.

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=& \frac{2}{e}\sqrt{\frac{1}{\pi\frac{m^2}{2}}} \cdot \frac{(m+2)^{m+3}}{\left(\frac{m}{2}\right)^{m}} \\
=& \frac{2\sqrt{2}}{e\sqrt{\pi}} \cdot \frac{(m+2)^{m+3}2^m}{m^{m+1}}
\end{aligned}$$

More Asymptotics (Recall: $\# = \left\lceil \frac{m+2-\sqrt{m+2}}{2} - 1 \right\rceil$)

Note: $\# \sim \frac{m}{2}$, $m - \# \sim \frac{m}{2}$, and $m - 2\# \sim \sqrt{m+2}$.

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=& \frac{2}{e}\sqrt{\frac{1}{\pi\frac{m^2}{2}}} \cdot \frac{(m+2)^{m+3}}{\left(\frac{m}{2}\right)^{m}} \\
=& \frac{2\sqrt{2}}{e\sqrt{\pi}} \cdot \frac{(m+2)^{m+3}2^m}{m^{m+1}} \\
=& \frac{2\sqrt{2}}{e\sqrt{\pi}} \cdot \left(\frac{m+2}{m}\right)^{m+1} (m+2)^2 \cdot 2^m
\end{aligned}$$

More Asymptotics (Recall: $\# = \left\lceil \frac{m+2-\sqrt{m+2}}{2} - 1 \right\rceil$)

Note: $\# \sim \frac{m}{2}$, $m - \# \sim \frac{m}{2}$, and $m - 2\# \sim \sqrt{m+2}$.

$$
\begin{aligned}
f(m) =& 2\sqrt{\frac{(m+2)}{2\pi\#(m-\#+1)}} \cdot \frac{(m+2)^{m+2}}{\#^{\#}(m-\#+1)^{m-\#+1}} \cdot \frac{1}{e} \cdot [m - 2\# + 1] \\
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=& \frac{2}{e}\sqrt{\frac{1}{\pi\frac{m^2}{2}}} \cdot \frac{(m+2)^{m+3}}{\left(\frac{m}{2}\right)^{m}} \\
=& \frac{2\sqrt{2}}{e\sqrt{\pi}} \cdot \frac{(m+2)^{m+3}2^m}{m^{m+1}} \\
=& \frac{2\sqrt{2}}{e\sqrt{\pi}} \cdot \left(\frac{m+2}{m}\right)^{m+1}(m+2)^2 \cdot 2^m = \boldsymbol{\frac{2e\sqrt{2}}{\sqrt{\pi}} \cdot 2^m m^2}.
\end{aligned}
$$

Recap

- The tensor product $\boldsymbol{a} \bigotimes \boldsymbol{b}$ is every entry in $\boldsymbol{a}$ multiplied by every entry in $\boldsymbol{b}$.
- The 1-norm of a vector is the sum of the absolute values of its coordinates.
- If we can relate something to a polynomial, it can be much easier to understand it!
- At the cost of some accuracy, asymptotics can give us a simpler form for a complicated-looking answer!

Thank You!

- Marist Mathematics Department
- Dr. Carl Wang-Erickson, University of Pittsburgh
- All of you, for your attention!